Cohomological duality of $\hat{\mathbb{Z}}$

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1 Introduction

Often in Galois cohomology, one needs to compute the cohomology groups of a $\hat{\mathbb{Z}}$-module. The two most common reasons for this are the cohomology of Galois modules over a finite field $\mathbb{F}_q$, and the unramified cohomology of a Galois module over K for a local field K.

It turns out that there exist very powerful theorems for computing these cohomology groups, varying from explicit formulas, duality theorems, and even class field theory-like results. Unfortunately, I am not aware of any particular source that accumulates all of these results in one place, so I have set out to change this. These notes are a collection of results I have seen in [NSW13, Ked21, Pap00], along with some generalizations I found by experimenting. (I make no claims that I am the first to come up with these generalizations.)

Before we begin, let us fix some notation: G we be a profinite group isomorphic to $\hat{\mathbb{Z}}$ unless stated otherwise. It has a canonical topological generator, ϕ. A is a discrete additive $\hat{\mathbb{Z}}$-module that will potentially have additional hypotheses on it. For a profinite group U, $H^i(U, A)$ denotes group cohomology of U acting on a discrete module A, and $\hat{H}^i(U, A)$ denotes the Tate groups.

The outline of sections is as follows: section 2 has an explicit determination of $H^1(G, A)$ for arbitrary A, section 3 is dedicated to the proof that $|H^0(G, A)| = |H^1(G, A)|$ for finite A. Section 4 shows that $\operatorname{cd} G = 1$ and $\operatorname{scd} G = 2$. Section 5 discussed duality results for torsion modules, and section 6 discusses a class field theory for G, including a duality result for $\mathbb{Z}$-free modules. Some exercises are given in section 7.

2 Some Explicit Calculations

Let A be a discrete G module. Our goal for this section is to obtain a convinient explicit formulas for $H^1(G, A)$. The upshot of this mission is that we already have a semi-useful explicit form of crossed homomorphisms modulo principal crossed homomorphisms. With that in mind, this section will basically amount to a determination of what these two types of objects "look like" for the case of $G = \hat{\mathbb{Z}}$

For $n \in \mathbb{N}$, we define a series of element $N_n \in \mathbb{Z}[G]$ by

$$N_n = \sum_{i=0}^{n-1} \phi^i,$$

and we extend these to $n \in \mathbb{Z}$ by $N_{n-1} = N_n - \phi^n$. These elements should be thought of as "partial norm elements." To make this intuition precise, consider for $n \in \mathbb{N}$ the map $\mathbb{Z}[G] \to \mathbb{Z}[G/G^n] = \mathbb{Z}[C_n]$. There is a true norm element of $\mathbb{Z}[C_n]$, and N_n is a preimage of it under this map.

Definition 1. *Let A be a discrete G-module. Define ${}_{N_G}A$ by*

$${}_{N_G}A = \{a \in A : N_n a = 0 \text{ for some } n\}.$$

We call ${}_{N_G}A$ the eventual norm kernel of A.

One can verify that if $N_n a = 0$, then $N_{mn} a = 0$ for any $m \in \mathbb{Z}$. As a consequence, we obtain the fact that ${}_{N_G}A$ is a $\mathbb{Z}[G]$-submodule of A. With this, we can state the main result of this section, which is a generalization of [Pap00, Thm. 1.1].

Proposition 1. *Let A be a discrete G module. Then there is a canonical isomorphism ev_ϕ (defined later) so that*

$$H^1(G,A) \cong {}_{N_G}A/(1-[\phi])A$$

Let $Z^1(G,A)$ denote the group of crossed homomorphisms (inhomogeneous 1-cocylces) and $B^1(G,A)$ denote the group of principal crossed homomorphisms (inhomogeneous 1-coboundaries). We recall that elements of $Z^1(G,A)$ are functions $f : G \to A$ that satisfy $f(gh) = f(g) + gf(h)$, and elements of $B^1(G,A)$ satisfy $f(g) = a - ga$ for some $a \in A$. To prove proposition 1, we will prove the following

Lemma 1. *Define $\mathrm{ev}_\phi : Z^1(G,A) \to A$ by $\mathrm{ev}_\phi(f) = f(\phi)$. Then ev_ϕ induces an isomorphism from $Z^1(G,A) \to {}_{N_G}A$.*

Proof. First we show that ev_ϕ is injective. Suppose $f \in \ker(\mathrm{ev}_\phi)$. Then $f(\phi) = 0$, and by the definition of crossed homomorphism we obtain $f(\phi^n) = 0$, and by continuity we obtain $f(g) = 0$ for all $g \in G$.

Now we show that $\mathrm{im}(\mathrm{ev}_\phi) = {}_{N_G}A$. First, suppose f is a cochain. By continuity, we obtain $f(\phi^n) = f(\phi^m)$ for positive integers $m > n$. Then, by the definition of crossed homomorphism and induction, we deduce that $f(\phi^n) = N_n f(\phi)$. Putting these together, we obtain

$$0 = f(\phi^m) - f(\phi^n) = (N_m - N_n)f(\phi) = \phi^n N_{m-n} f(\phi)$$

which implies $N_{m-n}f(\phi) = 0$, so $f(\phi) \in {}_{N_G}A$. This yields $\mathrm{im}(\mathrm{ev}_\phi) \subseteq {}_{N_G}A$.

Now we show the reverse inclusion. Let $a \in {}_{N_G}A$ and define $f_a : \langle\phi\rangle \to A$ by $f_a(\phi^n) = N_n a$, and by computation we verify the cocycle condition on f_a within its domain. We claim that f_a is continuous. Since A is discrete, there exists $n \in \mathbb{N}$ such that G^n acts trivially on a. Similarly, since $a \in {}_{N_G}A$ there exists $m \in \mathbb{N}$ such that $N_m a = 0$. A computation then shows that f_a factors through $\langle\phi^{\gcd(m,n)}\rangle$, so it is continuous. f_a then extends canonically via continuity to a cocycle $\tilde{f}_a : G \to A$. ☐

Now we can proceed to the proof of Proposition 1.

Proof of Proposition 1. With Lemma 1 in mind, we need only show that $B^1(G,A)$ is mapped to $(1-\phi)A$ under ev_ϕ. Suppose $\mathrm{ev}_\phi(f) \in (1-\phi)A$. Then $f(\phi) = (1-\phi)b$ for some $g \in G$. This agrees with the coboundary $\tilde{f}(g) = b - gb$, when evaluated at ϕ. Since ev_ϕ is injective, this implies $\tilde{f} = f$ and therefore $f \in B^1(G,A)$. On the other hand, if $f(g) = b - gb$ is a coboundary, then clearly $\mathrm{ev}_\phi(f) = b - \phi b \in (1-\phi)A$. ☐

Using this, we obtain the

Corollary 1. *If A is torsion, then $H^1(G,A) = A/(1-\phi)A$*

Proof. Using Proposition 1, it suffices to show that ${}_{N_G}A = A$. Let $a \in A$ be m_1-torsion. Since the action of G on A is continuous, a is fixed by G^{m_2} for some $m_2 \in \mathbb{Z}$. This means that action of $\mathbb{Z}[G]$ on a factors through the finite ring $(\mathbb{Z}/m_1\mathbb{Z})[G/G^{m_2}]$, so the $\mathbb{Z}[G]$-submodule of A generated by a is finite. This implies that $\{N_n a : n \in \mathbb{Z}\}$ is a finite set, so there exist $n_1 < n_2 \in \mathbb{Z}$ with $N_{n_1}a = N_{n_2}a$, which by computation yields $N_{n_2-n_1}a = 0$. ☐

We demonstrate these results with a nice number-theoretic application which can be found after the proof of Theorem 1.1 in [Pap00].

Proposition 2. *Let $E/\mathbb{F}_q$ be an Elliptic curve. Then $H^1(\mathbb{F}_q, E) = 0$*

Proof. The map $f : E(\bar{\mathbb{F}}_q) \to E(\bar{\mathbb{F}}_q)$ given by $f(P) = P - \phi P$ is a nonconstant algebraic map of curves. It is therefore surjective by [Sil09, Thm. II.2.3], so $E(\bar{\mathbb{F}}_q)/(1-\phi)E(\bar{\mathbb{F}}_q) = 0$. ☐

3 Euler Characteristic

For a moment, let G be an arbitrary profinite group that has finite Cohomological dimension. For a finite G-module A with $|H^i(G,A)| < \infty$, define the Euler characteristic

$$\chi(G,A) = \prod_{i=0}^{\infty} |H^i(G,A)|^{(-1)^i}.$$

Now let $G = \hat{\mathbb{Z}}$ once again, and let A be finite. In light of Theorem 1 and Corollary 1, we know that the conditions for $\chi(G,A)$ to make sense are satisfied. In this case, we have

$$\chi(G,A) = \frac{H^0(G,A)}{H^1(G,A)}.$$

Proposition 3. *Let A be a finite G-module, then $\chi(G,A) = 1$. Equivalently (by Theorem 1), $|H^0(G,A)| = |H^1(G,A)|$.*

Proof. Define $f : A \to A$ for $f(a) = (1-\phi)a$. Since G is procyclic, we have $\ker(f) = H^0(G,A)$, and by 1 we get $\operatorname{coker}(f) = H^1(G,A)$. By general properties of homomorphisms of finite groups, we have $|\operatorname{coker}(f)| = |\ker(f)|$, so $|H^0(G,A)| = |H^1(G,A)|$. □

4 Cohomological dimension

The main goal of this section is a discussion of the cohomological dimension of G. We recall the following definitions for cohomological dimension from [NSW13, Sec. 3.3.1]:

Definition 2. *Let G be a profinite group, we say that $\operatorname{cd} G = n$, if for all discrete torsion G-modules and $i > n$, $H^i(G,A) = 0$, and n is the minimal integer with this property.*

We say that $\operatorname{scd} G = n$ if the same conditions hold, but for all discrete G-modules, including ones that are not necessarily torsion.

Our goal will be to prove the following theorem

Theorem 1. *We have the identities $\operatorname{cd} G = 1$ and $\operatorname{scd} G = 2$.*

There is a common misconception that the cohomology of G should be periodic with period 2, since the cohomology of finite cyclic groups is periodic with period 2. Such a misconception is often accompanied with the following "proof."

By standard results, we have $H^n(G,A) = \varprojlim_m H^n(G/G^m, A)$ and $H^{n+2}(G,A) = \varprojlim_m H^n(G/G^m, A)$, with the inverse limit taken over inflation maps. Since $H^n(G/G^m, A) \cong H^{n+2}(G/G^m, A)$, it follows that $H^n(G,A) \cong H^{n+2}(G,A)$.

The problem with this argument is that while the groups in the inverse limit are isomorphic, the inflation maps are not the same. In particular, the following diagram (where the horizontal maps are periodicity isomorphisms) does not commute.

$$\begin{array}{ccc} H^n(G/G^{nm}, A) & \xrightarrow{\sim} & H^{n+2}(G/G^{nm}, A) \\ {\scriptstyle inf}\uparrow & & {\scriptstyle inf}\uparrow \\ H^n(G/G^m, A) & \xrightarrow{\sim} & H^{n+2}(G/G^m, A) \end{array}$$

The following proof is an elaboration on the one given as an example in [NSW13, Sec. 3.3].

Proof of Theorem 1. We first show $\operatorname{cd} G = 1$. First, note that $H^1(G, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z} \neq 0$, so $\operatorname{cd} G \geq 1$. We will show that if A is a finite torsion G module, $H^2(G,A) = 0$

By [NSW13, Thm. 1.2.4], it suffices to show that all exact sequences of the form

$$0 \to A \to \hat{G} \xrightarrow{\pi} G \to 0$$

split. By surjectivity of $\pi : \hat{G} \to G$, there exists a $\hat{\phi} \in \hat{G}$ such that $\pi(\hat{\phi}) = \phi$. Define a map $s' : \langle \phi \rangle \to \hat{G}$ by $s'(\phi^n) = \hat{\phi}^n$.

We claim that s' is continuous. First, since A is finite and G is profinite, $\hat{G}$ is profinite. We use the following result from the general theory of profinite groups: let $U \subseteq \hat{G}$ be open, and $u \in U$. Then there exists an $m \in \mathbb{N}$ such that $u\langle \hat{G}^m \rangle \subseteq U$, where $\langle \hat{G}^m \rangle$ is the subgroup of $\hat{G}$ generated by mth powers. (Exercise: prove this.)

Now we can show that s' is continuous: if U is an open set containing $s'(\phi^r)$, then U contains $s'(\phi^r)\langle \hat{G}^m \rangle$ for some m, and therefore the open subset $\phi^r\langle \phi^m \rangle \subseteq \langle \phi \rangle$ is contained in U, which means that s' is continuous.

Since s' is continuous, it extends to a continuous homomorphism $s : G \to \hat{G}$. Since $\pi \circ s$ is the identity when restricted to the dense subset $\langle \phi \rangle$, it is the identity on all of G and therefore the exact sequence splits.

We now know that $H^2(G, A) = 0$ when A is finite. Working one prime at a time and using the fact that all simple G modules are finite, we obtain $\operatorname{cd} G < 2$ by [NSW13, Prop. 3.3.2]. This implies $\operatorname{cd} G = 2$.

Now, by [NSW13, Prop. 3.3.3], we have $\operatorname{scd} G \leq \operatorname{cd} G + 1 = 2$. Since $H^2(G, \mathbb{Z}) \cong H^1(G, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$, we have $\operatorname{scd} G \geq 2$, so these together imply $\operatorname{scd} G = 2$. (Alternatively, one can use [NSW13, Sec. 3.3 Ex. 1]) ☐

5 Duality for Torsion Modules

Like the absolute Galois groups of local fields, the group G admits a duality theory that makes its cohomology especially nice. First we fix some notation, for an Abelian group or G-module A, write $A^* = \hom_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$. To establish this duality theory, we will need to define the following *Dualizing modules* (See [NSW13, Def. 2.5.1]).

$$D_i(A) = \varinjlim_{cor^*} H^i(U, A)^*$$

where the inverse limit is taken over open subgroups of G. We will abuse notation and write $D_i(\hat{\mathbb{Z}})$ for $\varinjlim D_i(\mathbb{Z}/m\mathbb{Z})$.

Lemma 2. *For all $m \in \mathbb{Z}$*

$$D_0(\mathbb{Z}/m\mathbb{Z}) = 0.$$

In our case, the open subgroups of G are precisely G^n for $n \in \mathbb{N}$.

Proof. We have the identies $H^0(G^n, \mathbb{Z}/m\mathbb{Z})^* = \mathbb{Z}/m\mathbb{Z}$. Since *cor* is the norm map in H^0, we have that $cor : H^0(G^{nn'}, \mathbb{Z}/m\mathbb{Z}) \to H^0(G^n, \mathbb{Z}/m\mathbb{Z})$ is given by mutliplication by n', and therefore its dual cor^* is also given by multiplication by n'. Computing the direct limit from this information we obtain $D_0(\mathbb{Z}/m\mathbb{Z}) = 0$ because every element is 0 far enough down the direct system. ☐

Lemma 3. *For all $m \in \mathbb{Z}$, we have*

$$D_1(\mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z},$$

moreover, $D_1(\hat{\mathbb{Z}}) = \mathbb{Q}/\mathbb{Z}$.

Proof. In this case, we once again have the identities $H^1(G^n, \mathbb{Z}/m\mathbb{Z}) = \hom(G^n, \mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$, so therefore $H^1(G^n, \mathbb{Z}/m\mathbb{Z})^* = \mathbb{Z}/m\mathbb{Z}$, however in this case it will turn out that cor^* is the identity.

To show this, note that $res : H^1(G^n, \mathbb{Z}/m\mathbb{Z}) \to H^1(G^{nn'}, \mathbb{Z}/m\mathbb{Z})$ is given by multiplication by n'. This is because, using the identification $H^1 = \hom$, restriction is literally restricting the domain. Since every element of $G^{nn'}$ is simply the n'th power of something in G^n, this makes *res* just the multiplication by n' map. Standard results tell us $cor \circ res$ is given by multiplication by n', so this implies *cor* is the identity, and therefore cor^* is the identity. This yields $D_1(\mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$.

To obtain $D_1(\hat{\mathbb{Z}}) = \mathbb{Q}/\mathbb{Z}$, we use the fact that the map $H^1(G^n, \mathbb{Z}/(mm')\mathbb{Z}) \to H^1(G^n, \mathbb{Z}/m\mathbb{Z})$ is given by projection, so its dual is given by the inclusion $\mathbb{Z}/m\mathbb{Z} \to m'\mathbb{Z}/mm'\mathbb{Z}$. Passing to the direct limit, the maps $D_1(\mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/mm'\mathbb{Z} = D_1(\mathbb{Z}/mm'\mathbb{Z})$ is given by inclusion, so the direct limit $D_1(\hat{\mathbb{Z}})$ is $\mathbb{Q}/\mathbb{Z}$. ☐

With these results, we have the following duality theorem

Theorem 2. *Let A be a discrete torsion G-module. Then the cup product induces a perfect pairing*

$$\cup : H^0(G, A) \times H^1(G, A^*) \to H^1(G, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}.$$

In particular, we have isomorphisms

$$H^0(G, A)^* \cong H^1(G, A^*).$$

Proof. The following facts are true:

1. $\operatorname{cd} G = 1$ (Theorem 1.)
2. $D_0(\mathbb{Z}/p\mathbb{Z}) = 0$ for all p (Lemma 2.)

Then by [NSW13, Thm. 3.4.6], the cup product induces a perfect pairing

$$\cup : H^i(G, \hom(A, D_1(\hat{\mathbb{Z}}))) \times H^{1-i}(G, A) \to H^1(G, D_1(\hat{\mathbb{Z}})) = \mathbb{Q}/\mathbb{Z}.$$

Since $D_1(\hat{\mathbb{Z}}) = \mathbb{Q}/\mathbb{Z}$ by Lemma 3, we obtain the result. □

We could have actually proven the isomorphism $H^0(G, A)^* \cong H^1(G, A^*)$ using the morphism f from the proof of 3, however this method would not show that the isomorphism is induced by the cup product. We leave this as an exercise.

6 Class Field Theory

Let G be an arbitrary profinite group for a moment. If $\operatorname{scd} G = 2$, then we should expect G to "have a class field theory." This is obviously a vaguely hand-wavey notion, but it can be made rigorous (see [NSW13, Th. 3.4.6]). As we have shown this property for $G = \hat{\mathbb{Z}}$ in Theorem 1, we should expect this property for $G = \hat{\mathbb{Z}}$. To this end, recall the following definition of a formation module from [NSW13, Def. 3.1.8]

Definition 3. *Let G be a profinite group. A formation module for G is a discrete G module C together with a system of isomorphisms*

$$inv_{U/V} : H^2(U/V, C^V) \xrightarrow{\sim} \frac{1}{[U:V]}\mathbb{Z}/\mathbb{Z}$$

for every pair $V \subseteq U$ of open subgroups with V normal in U. We further require:

1. *$H^1(U/V, C^V) = 0$*
2. *For open normal subgroups $W \subseteq V$ of the open subgroup U, the following diagram commutes.*

$$\begin{array}{ccccc} H^2(U/V, C^V) & \xrightarrow{inf} & H^2(U/W, C^W) & \xrightarrow{res} & H^2(V/W, C^W) \\ \downarrow{\scriptstyle inv} & & \downarrow{\scriptstyle inv} & & \downarrow{\scriptstyle inv} \\ \frac{1}{[U:V]}\mathbb{Z}/\mathbb{Z} & \xhookrightarrow{inc} & \frac{1}{[U:W]}\mathbb{Z}/\mathbb{Z} & \xrightarrow{[U:V]} & \frac{1}{[V:W]}\mathbb{Z}/\mathbb{Z} \end{array}$$

We remark that an inverse limit argument implies that we have $H^1(G, C) = 0$ and $H^2(G, C) = \frac{1}{\#G}\mathbb{Z}/\mathbb{Z}$, and we call the second isomorphism inv. The main (interesting) consenquence of the existence of a formation module is the following duality theorem.

Theorem 3 ([NSW13] Thm. 3.1.9)**.** *Let G be a profinite group and C a formation module for G. If A is a discrete G-module which is finitely generated and free as a $\mathbb{Z}$-module. Then for all $i \in \mathbb{Z}$, the cup product gives a perfect pairing*

$$\cup : \hat{H}^i(G, \hom(A, C)) \times \hat{H}^{2-i}(G, A) \to H^2(G, C) \xrightarrow{inv} \frac{1}{\#G}\mathbb{Z}/\mathbb{Z}$$

which induces a topological isomorphism

$$\hat{H}^i(G, \hom(A, C)) \cong \hat{H}^{2-i}(G, A)^*.$$

The groups $\hat{H}^i(G,A)$ for profinite G are slightly complicated, but for our sake you need only know that

$$\hat{H}^0(G,A) = \varprojlim_U \hat{H}^0(G/U, A^U).$$

We now return to the case of $G = \hat{\mathbb{Z}}$ use these theorems, we must construct a formation module for G. To this end we have the following lemma, which is based on [Ked21, Ex. 5.1.1]

Lemma 4. *The module $C = \mathbb{Z}$ is a formation module for G.*

Proof. We fix the following notation, all extending the notation established in 3. Let $U = G^\ell$ and $V = G^m$. We first verify condition (1) since

$$H^1(U/V, C^V) = H^1(m\mathbb{Z}/\ell\mathbb{Z}, \mathbb{Z}) = 0.$$

Next we verify condition (2). We consider the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

Taking U/V cohomology, we see that $H^i(U/V, \mathbb{Q}) = 0$ because $\mathbb{Q}$ is uniquely divisible, so we get an isomorphism $\delta^{-1} : H^2(U/V, \mathbb{Z}) \to H^1(U/V, \mathbb{Q}/\mathbb{Z})$. Now,

$$H^1(U/V, \mathbb{Q}/\mathbb{Z}) = \hom(G^m/G^\ell, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \frac{1}{[U:V]}\mathbb{Z}/\mathbb{Z}.$$

We make a specific concrete choice for the last isomorphism, which we will call it ψ. Let $f \in \hom(G^m/G^\ell, \mathbb{Q}/\mathbb{Z})$ be the morphism that takes ϕ^m to m/ℓ, then define $\psi(f) = m/\ell$. (ψ can be thought of as evaluation at ϕ^m.)

Verifying the commutativity of the diagram in (2) is a routine but slightly tedious calculation that we leave an an exercise. It mainly comes down to using an explicit description of inf and res on cocycles, which for example can be found in [NSW13, Sec. 1.5]. □

There is another choice for C, and that is $\hat{\mathbb{Z}}$. This result follows from the fact that $\hat{\mathbb{Z}}$ is "Cohomologically identical" to $\mathbb{Z}$. If G is any profinite group, then for all $i > 0$, we have $H^i(G,\mathbb{Z}) \cong H^i(G,\hat{\mathbb{Z}})$, which can be proven using the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \hat{\mathbb{Z}} \longrightarrow \hat{\mathbb{Z}}/\mathbb{Z} \longrightarrow 0$$

and the fact that $\hat{\mathbb{Z}}/\mathbb{Z}$ is uniquely divisible.

Using $\hat{\mathbb{Z}}$ has the upside of being a level-compact formation module, but this does not actually give us much of an advantage, so we have chosen to stick with $\mathbb{Z}$.

The existence of a formation module has many consequences, however most of them are either trivial or already proven through other means. The one interesting result we get is a duality theorem: applying Theorem 3 we obtain

Proposition 4. *Let A be a finitely generated $\mathbb{Z}$-free G module. Then the cup product induces isomorphisms*

$$\hat{H}^i(G, \hom(A,\mathbb{Z})) \cong \hat{H}^{2-i}(G,A)^*.$$

7 Exercises

1. Determine the periodicity isomorphism between $\hat{H}^0(G/G^n, \mathbb{Z}/m\mathbb{Z})$ and $\hat{H}^2(G/G^n, \mathbb{Z}/m\mathbb{Z})$, along with the inflation maps, and demonstrate that the diagram on page 3 does not commute.

2. Let G be a profinite group and $u \in U \subseteq G$ be an open subset. Show that there exists $m \in \mathbb{N}$ such that $u\langle G^m \rangle \subseteq U$, where $\langle G^m \rangle$ is the subgroup of G generated by its mth powers.

3. Using Corollary 1 and the ideas in section 3, show that $H^0(G,A) \cong H^1(G,A^*)$ for torsion A without using dualizing modules.

4. Verify the commuting diagram in the proof of Lemma 4.
5. Let A be finitely generated as an abelian group, A_{tors} its torsion subgroup, and $A_{\text{free}} = A/A_{\text{tors}}$. Prove that $H^2(G, A) = H^2(G, A_{\text{free}})$ and the exact sequence

$$0 \longrightarrow A_{\text{tors}}/((1-\phi)A \cap A_{\text{tors}}) \longrightarrow H^1(G, A) \longrightarrow H^1(G, A_{\text{free}}) \longrightarrow 0$$

6. $\hat{\mathbb{Z}}$ is not the only profinite group with particularly simple cohomology. In fact, it turns out that every torsion-free procyclic group has especially nice cohomology.[1] Specifically, let $P \subseteq \{2, 3, 5, \ldots\} = P_{\max}$ be a set of rational primes. Define

$$\mathbb{Z}_P = \prod_{p \in P} \mathbb{Z}_p,$$

and note that $\hat{\mathbb{Z}} = \mathbb{Z}_{P_{\max}}$. Generalize the theorems here to $\mathbb{Z}_P$ for arbitrary $P \subseteq P_{\max}$

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[1] Handling procyclic groups with a torsion part can be done by combining the results in this PDF with standard results on the cohomology of finite cyclic groups.